MATH 1272: Calculus II

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12.3 - Dot Product

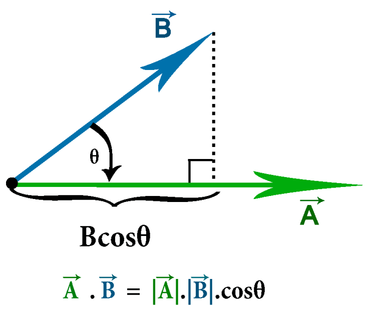
Review:

Dot Product: If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then the dot product of $\vec{a}$ and $\vec{b}$ is the number $\vec{a} \cdot \vec{b}$ or $\langle \vec{a}, \vec{b} \rangle$ given by $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$. Observe that with this type of "multiplication," we start with two vectors, but end up with a scalar. As a result, this is also called the **scalar product**, or inner product. In the next section, we will learn of a different type of vector multiplication where the result is a vector (vector product or cross product).

Properties of the Dot Product

- $\vec{a} \cdot \vec{a} = |\vec{a}|^2 = \left(\sqrt{a_1^2 + \ldots + a_n^2}\right)^2 = a_1^2 + \ldots + a_n^2$,
- $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ Commutativity.
- $\vec{a} \cdot \left(\vec{b} + \vec{c}\right) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$ Distributivity.
- $\left(c\vec{a}\right) \cdot \vec{b} = c\left(\vec{a} \cdot \vec{b}\right) = \vec{a} \cdot \left(c\vec{b}\right)$ Scalar Associativity.
- $\vec{0} \cdot \vec{a} = 0$ Scalar Zero.

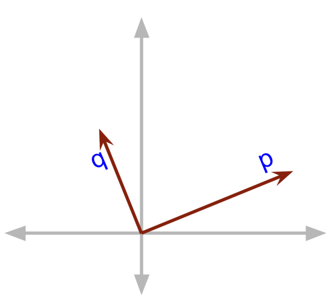
Dot Product Angle Theorem: If θ is the angle between $\vec{a}$ and $\vec{b}$ where $0 \leq \theta \leq \pi$, then $\vec{a} \cdot \vec{b} = |\vec{a}|\left|\vec{b}\right| \cos\theta$.



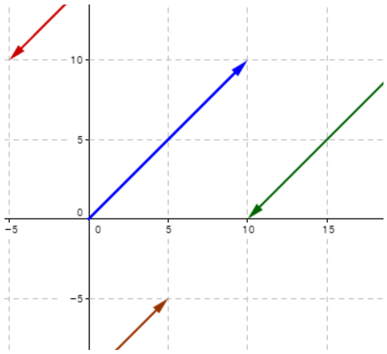
Corollary: Let θ be the angle between the (nonzero) vectors $\vec{a}$ and $\vec{b}$, then: $\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\left|\vec{b}\right|}$.

Perpendicular/Orthogonal: Say we have nonzero vectors $\vec{a}$ and $\vec{b}$ where the acute angle between them is $\theta = \frac{\pi}{2}$. In this case, the previous theorem gives us $\vec{a} \cdot \vec{b} = |\vec{a}|\left|\vec{b}\right| \cos\frac{\pi}{2} = 0$. Conversely, if $\vec{a} \cdot \vec{b} = 0$, then $\cos\theta = 0$ and the acute angle θ must be equal to $\frac{\pi}{2}$. The zero vector $\vec{0}$ is considered to be perpendicular to all vectors, therefore we have:

Vector Orthogonality Criteria: Two vectors $\vec{a}$ and $\vec{b}$ are orthogonal if and only if $\vec{a} \cdot \vec{b} = 0$.



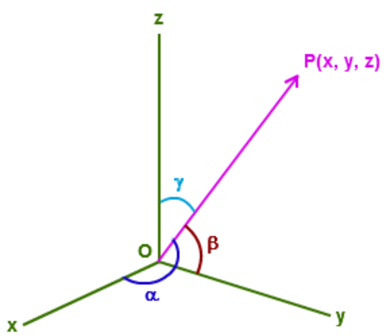
Parallel: Also observe that $\cos\theta = \pm 1$ when $\vec{a}$ and $\vec{b}$ are parallel to each other. Therefore, we can determine whether to vectors are parallel by looking at whether $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \pm 1$.



Useful Tidbit: You can easily experiment to find that $\vec{a}\cdot\vec{b}$ is positive for $\theta < \frac{\pi}{2}$, and negative for $\theta > \frac{\pi}{2}$. We can therefore think of $\vec{a}\cdot\vec{b}$ as measuring the extent to which the two vectors point in the same direction: positive when in the same direction, 0 when perpendicular, and negative when pointing in the opposite direction.

Direction Angles and Direction Cosines

Let $\vec{P}$ be a nonzero vector in three dimensions.
Direction Angles: The angles $\alpha,\ \beta,$ and γ are those angles (all in the interval $[0,\ \pi]$) that the vector $\vec{P}$ makes with the positive $x,\ y,$ and z-axes, respectively.

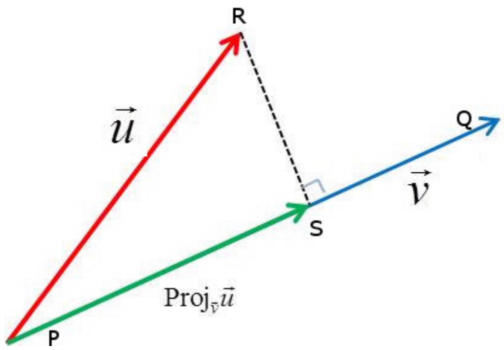


Direction Cosines: $\cos\alpha,\ \ \cos\beta,\ \ \cos\gamma$.
Simple calculations reveal that using these angles, we can rewrite $\vec{P}$ as
$\vec{P} = \langle x, y, z\rangle = \left\langle \left|\vec{P}\right|\cos\alpha,\ \left|\vec{P}\right|\cos\beta,\ \left|\vec{P}\right|\cos\gamma \right\rangle = \left|\vec{P}\right|\left\langle \cos\alpha,\ \cos\beta,\ \cos\gamma \right\rangle$.
Therefore, the unit vector is: $\frac{\vec{P}}{|\vec{P}|} = \left\langle \cos\alpha,\ \cos\beta,\ \cos\gamma \right\rangle$.

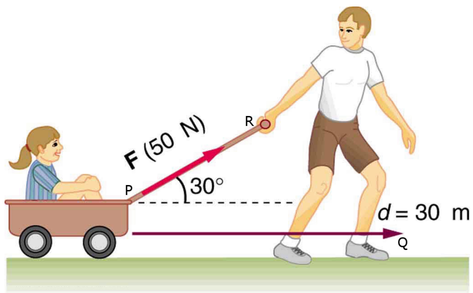
Projections

Vector Projection: If S is the foot of the perpendicular from R to the line containing $\overrightarrow{PQ}$, then the vector with representation $\overrightarrow{PS}$ is called the vector projection of $\vec{u}$ onto $\vec{v}$ and is denoted by $proj_v\vec{u}$ (also thought of as the "shadow" of $\vec{u}$).
So we have:

Vector Projection of $\vec{u}$ onto $\vec{v}$: $\quad proj_v\vec{u} = \left(\frac{\vec{v}\cdot\vec{u}}{|\vec{v}|}\right)\frac{\vec{v}}{|\vec{v}|} = \frac{\vec{v}\cdot\vec{u}}{|\vec{v}|^2}\vec{v}$.



Scalar Projection of $\vec{u}$ onto $\vec{v}$: This is the signed magnitude (scalar component) of the vector projection, which is the number $|\vec{u}|\cos\theta$, where θ is the angle between the vectors, denoted $comp_v\vec{u}$. So we have:
Scalar Projection of $\vec{u}$ onto $\vec{v}$: $\quad comp_v\vec{u} = \frac{\vec{v}\cdot\vec{u}}{|\vec{v}|}$,



Displacement Vector: Suppose that some constant force is written as a vector $\vec{F} = \overrightarrow{PR}$. And suppose this force is moving an object from P, but in a direction not parallel to $\overrightarrow{PR}$ (see image above where the object is moved along the ground, instead of in the direction of the force). If the force moves the object from P to Q, then the displacement vector is notated $\vec{D} := \overrightarrow{PQ}$.

Work: This is the product of the component of the force along $\vec{D}$ ($|\vec{F}|\cos\theta$), by the distance moved ($|\vec{D}|$) or: $W = \left(|\vec{F}|\cos\theta\right)|\vec{D}|$. Therefore, using the dot product angle theorem, we have $W = |\vec{F}||\vec{D}|\cos\theta = \vec{F}\cdot\vec{D}$.

Problem #4 Letting $\vec{a} = \langle 6,-2,3\rangle$ and $\vec{b} = \langle 2,5,-1\rangle$, find $\vec{a}\cdot\vec{b}$.

$\vec{a}\cdot\vec{b} = 6\cdot 2 + (-2)\cdot 5 + 3\cdot(-1) = -1.$

Problem #18 Find the angle between the vectors $\vec{a} = \langle 4,0,2\rangle$, and $\vec{b} = \langle 2,-1,0\rangle$. (First find an exact expression and then approximate to the nearest degree.)

$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

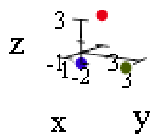
$|\vec{a}| = \sqrt{4^2+0^2+2^2} = 2\sqrt{5}, \qquad |\vec{b}| = \sqrt{2^2+(-1)^2+0^2} = \sqrt{5}.$

$\vec{a}\cdot\vec{b} = \langle 4,0,2\rangle\cdot\langle 2,-1,0\rangle = 4\cdot 2 + 0\cdot(-1) + 2\cdot 0 = 8.$

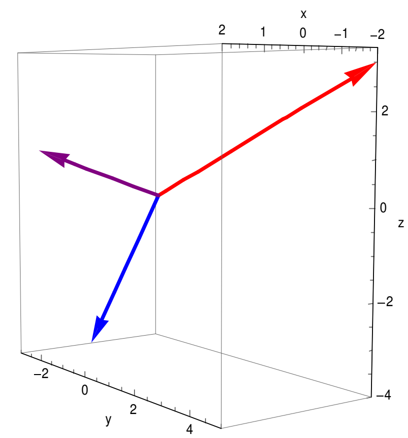
$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{8}{2\sqrt{5}(\sqrt{5})} = \frac{4}{5}.$

$\theta = \cos^{-1}\frac{4}{5} \approx 37^\circ$

Problem #22 Find, correct to the nearest degree, the three angles of the triangle with the vertices $A(1,0,-1),\ B(3,-2,0),\ C(1,3,3)$.



$\overrightarrow{AB} = (3-1, -2-0, 0+1) = (2, -2, 1)$,
$\overrightarrow{BC} = (1-3, 3+2, 3-0) = (-2, 5, 3)$,
$\overrightarrow{CA} = (1-1, 0-3, -1-3) = (0, -3, -4)$.



"**If** θ **is the angle between** $\vec{a}$ **and** $\vec{b}$ **where** $0 \le \theta \le \pi$, **then** $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$."

$$\cos\theta_B = \frac{\overrightarrow{AB}\cdot\overrightarrow{BC}}{|\overrightarrow{AB}||\overrightarrow{BC}|} = \frac{-2\cdot 2-2\cdot 5+1\cdot 3}{\sqrt{2^2+2^2+1^2}\sqrt{2^2+5^2+3^2}} = \frac{-4-10+3}{\sqrt{9}\sqrt{38}} = \frac{-11}{3\sqrt{38}} \approx -0.595,$$

$$\cos\theta_C = \frac{\overrightarrow{BC}\cdot\overrightarrow{CA}}{|\overrightarrow{BC}||\overrightarrow{CA}|} = \frac{-2\cdot 0-5\cdot 3-3\cdot 4}{\sqrt{2^2+5^2+3^2}\sqrt{0^2+3^2+4^2}} = \frac{-15-12}{\sqrt{38}\sqrt{25}} = -\frac{27}{5\sqrt{38}} \approx -0.876,$$

$$\cos\theta_A = \frac{\overrightarrow{CA}\cdot\overrightarrow{AB}}{|\overrightarrow{CA}||\overrightarrow{AB}|} = \frac{2\cdot 0+3\cdot 2-4\cdot 1}{\sqrt{0^2+3^2+4^2}\sqrt{2^2+2^2+1^2}} = \frac{6-4}{\sqrt{25}\sqrt{9}} = \frac{2}{15} \approx 0.133.$$

Therefore, $\theta_B = \cos^{-1}\left(-\frac{11}{3\sqrt{38}}\right) \approx 2.208$,
Or, $\cos^{-1}\left(\frac{11}{3\sqrt{38}}\right) \approx 0.9338$,
$\theta_C = \cos^{-1}\left(-\frac{27}{5\sqrt{38}}\right) \approx 2.638$
Or, $\cos^{-1}\left(\frac{27}{5\sqrt{38}}\right) \approx 0.5033$,
$\theta_A = \cos^{-1}\left(\frac{2}{15}\right) \approx 1.437$
Or, $\cos^{-1}\left(-\frac{2}{15}\right) \approx 1.705$.

We need $\theta_A + \theta_B + \theta_C \approx \pi$.

Therefore, $\theta_B \approx 0.9338$, $\theta_C \approx 0.5033$, $\theta_A \approx 1.705$.

Problem #24 Determine whether the following vectors are orthogonal, parallel, or neither.
a) $\vec{u} = \langle -3, 9, 6\rangle$, $\vec{v} = \langle 4, -12, -8\rangle$.

$\vec{u} \cdot \vec{v} = -3 \cdot 4 + 9 \cdot (-12) + 6 \cdot (-8) = -168 \neq 0.$ Not orthogonal.
$|\vec{u}| = \sqrt{3^2 + 9^2 + 6^2} = 3\sqrt{14}, \quad |\vec{v}| = \sqrt{4^2 + 12^2 + 8^2} = 4\sqrt{14}.$

Therefore, $\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \frac{-168}{3\sqrt{14} \cdot 4\sqrt{14}} = -1.$

So, $\vec{u}$ and $\vec{v}$ are parallel.

b) $\vec{u} = \vec{i} - \vec{j} + 2\vec{k}, \quad \vec{v} = 2\vec{i} - \vec{j} + \vec{k}.$

$\vec{u} \cdot \vec{v} = 1 \cdot 2 - 1 \cdot (-1) + 2 \cdot 1 = 5$

$\neq 0.$ Not orthogonal.

$|\vec{u}| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}. \quad |\vec{v}| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}.$

Therefore, $\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \frac{5}{6} \neq \pm 1.$ So, $\vec{u}$ and $\vec{v}$ are NOT parallel.

c) $\vec{u} = \langle a, b, c \rangle, \quad \vec{v} = \langle -b, a, 0 \rangle.$

$\vec{u} \cdot \vec{v} = a \cdot (-b) + b \cdot a + c \cdot 0 = 0.$

Orthogonal, so not parallel.

Problem #30 Find the acute angle between the lines $x + 2y = 7$ and $5x - y = 2$.

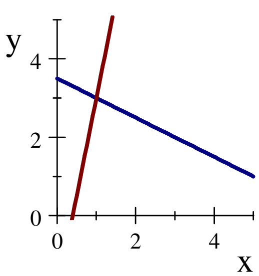
For the first line, we have points $(7,0)$ and $\left(0, \frac{7}{2}\right)$. For the second line we have points $(\frac{2}{5}, 0)$ and $(0, -2)$.

Therefore, we can define vectors $\vec{u} = 2 \cdot \left\langle 7 - 0, 0 - \frac{7}{2} \right\rangle = \langle 14, -7 \rangle$ and $\vec{v} = 5 \cdot \langle \frac{2}{5} - 0, 0 + 2 \rangle = \langle 2, 10 \rangle$, for the two lines respectively.

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \frac{14 \cdot 2 + (-7) \cdot 10}{\sqrt{14^2 + 7^2}\sqrt{2^2 + 10^2}} = \frac{28 - 70}{\sqrt{245}\sqrt{104}} = \frac{-42}{7\sqrt{5} \cdot 2\sqrt{26}} = \frac{-3}{\sqrt{130}}.$$

Therefore, $\theta_1 = \cos^{-1}\left(\frac{-3}{\sqrt{130}}\right) \approx 1.8375 > \frac{\pi}{2} \approx 1.57,$ so this is the obtuse angle.

So the acute angle must be $\theta_2 = \pi - \cos^{-1}\left(\frac{-3}{\sqrt{130}}\right) \approx 1.3045.$



Problem #42 Given $\vec{a} = \langle -2, 3, -6 \rangle$ and $\vec{b} = \langle 5, -1, 4 \rangle$, find the scalar and vector projections of $\vec{b}$ onto $\vec{a}$.

Scalar Projection of $\vec{a}$ onto $\vec{b}$:

$$comp_b\vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \frac{-2 \cdot 5 + 3(-1) - 6(4)}{\sqrt{2^2 + 3^2 + 6^2}} = \frac{-10 - 3 - 24}{\sqrt{49}} = \frac{-37}{7}.$$

Vector Projection of $\vec{a}$ onto $\vec{b}$:

$proj_b \vec{a} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}\right) \frac{\vec{b}}{|\vec{b}|} = \frac{-37}{7} \frac{\langle 5,-1,4\rangle}{\sqrt{5^2+1^2+4^2}} = -\frac{37}{7\sqrt{42}} \langle 5,-1,4\rangle.$

Problem #50 A tow truck drags a stalled car along the road. The chain makes an angle of 30° with the road and the tension in the chain is 1500 N. How much work is done by the truck in pulling the car 1 km ?

$W = \vec{F} \cdot \vec{D} = |\vec{F}||\vec{D}| \cos\theta.$

Observe that the units for Newtons are $\frac{kg \cdot m}{s^2}$, and the units for work (Joules) are $\frac{kg \cdot m^2}{s^2}$.

Therefore, to maintain consistent units, we convert the distance from kilometers to meters.

$W = 1500 \cdot 1000 \cdot \cos \frac{\pi}{6} \approx 1,299,038 \ J.$